HOSSAM GHANEM

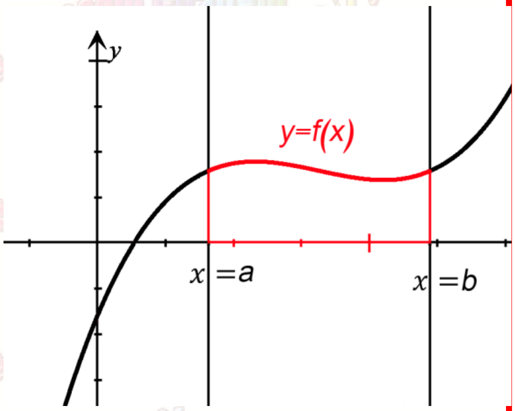
(32) 9.2 Area Of A Surface Of Revolution

The surface area of the surface obtained by rotating the curve $y = f(x)$ $\;a \le x \le b$ about $x - axis$

$$s = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Or

$$s = 2\pi \int_c^d y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$



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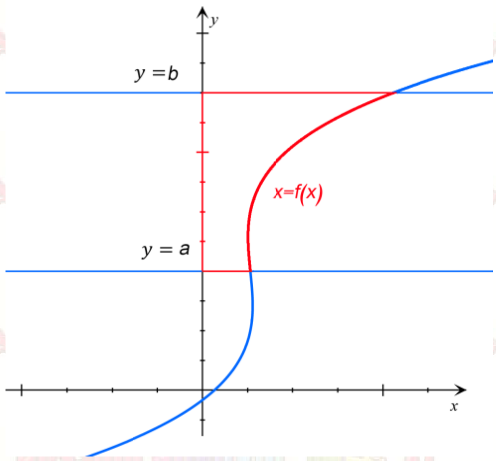

Example 1

29 January 2007 A

Let the curve C : $y = \frac{1}{8}x^2 - \ln x$, $x \in [\,1\,,2\,]$

Find the area of the surface of revolution obtained by rotating C about y-axis

Solution

$$y = \frac{1}{8}x^2 - \ln x$$

$$\frac{dy}{dx} = \frac{1}{4}x - \frac{1}{x}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{4}x\right)^2 + \left(\frac{1}{x}\right)^2 - \frac{1}{2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{4}x\right)^2 + \left(\frac{1}{x}\right)^2 + \frac{1}{2} = \left(\frac{1}{4}x + \frac{1}{x}\right)^2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \left(\frac{1}{4}x + \frac{1}{x}\right)$$

$$S = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_1^2 x\left(\frac{1}{4}x + \frac{1}{x}\right) dx$$

$$= 2\pi \int_1^2 \left(\frac{1}{4}x^2 + 1\right) dx = 2\pi \left[\frac{1}{12}x^3 + x\right]_1^2 = 2\pi \left[\frac{8}{12} + 2 - \left(\frac{1}{12} + 1\right)\right] = 2\pi \left[\frac{8}{12} + 2 - \frac{1}{12} - 1\right]$$

$$= 2\pi \left[\frac{8}{12} + 1 - \frac{1}{12}\right] = 2\pi \left(\frac{8 + 12 - 1}{12}\right) = \frac{19\pi}{6}$$

Example 2

27 June 2006 A

Find the area of the surface obtained by rotating the curve

$y = \sqrt{x + 1}$, $1 \le x \le 5$ about the $x-$axis.

Solution

$$y = \sqrt{x + 1}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x + 1}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4(x + 1)}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4(x + 1)} = \frac{4x + 4 + 1}{4(x + 1)} = \frac{4x + 5}{4(x + 1)}$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_1^5 \sqrt{x + 1} \cdot \frac{\sqrt{4x + 5}}{2\sqrt{x + 1}}\, dx = \pi \int_1^5 (4x + 5)^{\frac{1}{2}}\, dx = \pi \cdot \frac{2}{3} \cdot \frac{1}{4}\left[(4x + 5)^{\frac{3}{2}}\right]_1^5$$

$$= \frac{\pi}{6}\,[125 - 27] = \frac{98}{6}\pi = \frac{49}{3}\pi$$

Example 3

24 May 2005 A

Find the surface area of the solid generated by rotating the curve

$y = \cosh x \,, \quad 0 < x < 3$ about the $x -$ axis

Solution

$$y = \cosh x$$

$$\frac{dy}{dx} = \sinh x$$

$$\left(\frac{dy}{dx}\right)^2 = \sinh^2 x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 x = \cosh^2 x$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^3 \cosh x \cdot \cosh x \, dx = 2\pi \int_0^3 \cosh^2 x \, dx = 2\pi \int_0^3 \frac{1}{2}(\cosh 2x + 1)\, dx$$

$$= \pi \left[\frac{1}{2}\sinh 2x + x\right]_0^3 = \pi\left(\frac{1}{2}\sinh 6 + 3\right) - 0 = \pi\left(\frac{1}{2}\sinh 6 + 3\right)$$

Example 4

31 August 2008 A

The curve $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$, $2 \le x \le 4$, is rotated about the x-axis.

Find the area of the resulting surface.

Solution

$$y = \frac{x^2}{2} - \frac{\ln x}{4}$$

$$\frac{dy}{dx} = x - \frac{1}{4x}$$

$$\left(\frac{dy}{dx}\right)^2 = (x)^2 + \left(\frac{1}{4x}\right)^2 - \frac{1}{2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = (x)^2 + \left(\frac{1}{4x}\right)^2 + \frac{1}{2} = \left(x + \frac{1}{4x}\right)^2$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_2^4 \left(\frac{x^2}{2} - \frac{\ln x}{4}\right)\left(x + \frac{1}{4x}\right) dx = 2\pi \int_2^4 \left(\frac{1}{2}x^3 + \frac{1}{8}x - \frac{1}{4}x \ln x - \frac{1}{16}\cdot\frac{1}{x}\ln x\right) dx$$

$$I_1 = \int x \ln x \, dx$$

$u = \ln x \qquad dv = x\, dx$

$du = \frac{1}{x} dx \qquad v = \frac{1}{2}x^2$

$$I_1 = uv - \int v\, du$$

$$I_1 = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C_1$$

$$I_2 = \int \frac{1}{x}\ln x \, dx$$

$t = \ln x \qquad dt = \frac{1}{x} dx$

$$I_2 = \int t\, dt = \frac{1}{2}t^2 + C_2 = \frac{1}{2}(\ln x)^2 + C_2$$

$$S = 2\pi\left[\frac{1}{8}x^4 + \frac{1}{16}x^2 - \frac{1}{4}\left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right) - \frac{1}{16}\left(\frac{1}{2}(\ln x)^2\right)\right]_2^4 = 2\pi\left[\frac{1}{8}x^4 + \frac{1}{16}x^2 - \frac{1}{8}x^2 \ln x + \frac{1}{16}x^2 - \frac{1}{32}(\ln x)^2\right]_2^4$$

$$S = 2\pi\left[\frac{4^4}{8} + \frac{16}{16} - 2\ln 4 + \frac{16}{16} + \frac{1}{32}(\ln 4)^2 - \left(\frac{16}{8} + \frac{4}{16} - \frac{1}{2}\ln 2 + \frac{1}{4} + \frac{1}{32}(\ln 2)^2\right)\right]$$

Example 5

Let the curve C : $y = \frac{1}{8}x^4 + \frac{1}{4x^2}$, $x \in [\,1\,,2\,]$

Find the area of the surface of revolution obtained by rotating C about y-axis

Solution

$$y = \frac{1}{8}x^4 + \frac{1}{4x^2}$$

$$\frac{dy}{dx} = \frac{1}{2}x^3 - \frac{1}{2x^3}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x^3\right)^2 + \left(\frac{1}{2x^3}\right)^2 - \frac{1}{2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x^3\right)^2 + \left(\frac{1}{2x^3}\right)^2 + \frac{1}{2} = \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)^2$$

$$S = 2\pi\int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\;dx = 2\pi\int_1^2 x\left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)dx = 2\pi\int_1^2 \left(\frac{1}{2}x^4 + \frac{1}{2}x^{-2}\right)\;dx$$

$$= \pi\left[\frac{1}{10}x^5 - \frac{1}{2}x^{-1}\right]_1^2 = \pi\left[\frac{1}{10}x^5 - \frac{1}{2x}\right]_1^2 = \pi\left[\frac{32}{10} - \frac{1}{4} - \left(\frac{1}{10} - \frac{1}{2}\right)\right]\pi\left[\frac{32}{10} - \frac{1}{4} - \frac{1}{10} + \frac{1}{2}\right]$$

$$= \pi\left(\frac{64 - 5 - 2 + 10}{20}\right) = \frac{67}{10}\pi$$

Homework

1	Find the area of the surface obtained by rotating the curve $y = \cosh x\,,\ 0 \le x \le \ln\sqrt{2}$ about the $x -$ axis (4 pts.)	35 January 24, 2010
2	(4 pts.) Find the area of the surface obtained by rotating the curve $x = \sqrt{8y}$ For $y \in [\,0\,,1\,]$ about the $y -$ axis	36 June 6, 2010
3	(4 pts) Find the area of the surface obtained by rotating the curve $y = \cosh(x)$ $x \in [\,0\,,1\,]$ around the $y -$ axis.	40 August 7 , 2011